St George Girls High School

Year 11 – Higher School Certificate Course

Assessment Task 1

2006

Mathematics Extension 1

Time Allowed: 75 Minutes
(plus 5 minutes reading time)

Instructions to Candidates

1. Write using black or blue pen.
2. Attempt all questions.
3. Start each question on a new page.
4. Show all necessary working.
5. Marks for each question are shown in the right column.
6. Complete cover sheet clearly showing:
   - your name
   - your mathematics class and teacher.

Question 1 – (14 marks) – Start a New Page **Marks**

a) Differentiate with respect to x

(i) $y = \log_e(7x+6)$ 1

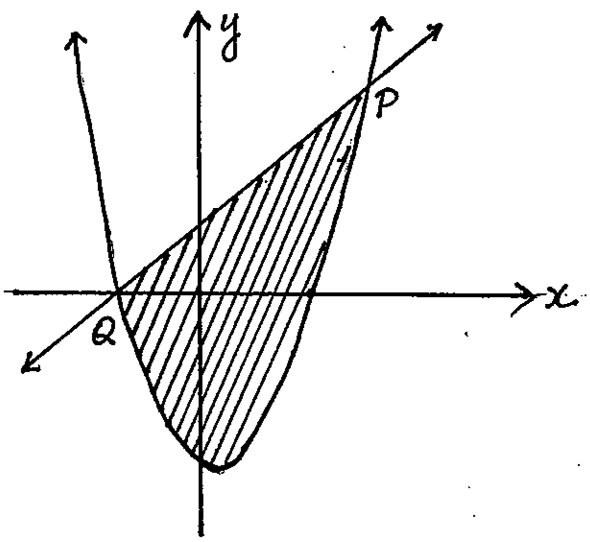
(ii) $y = e^{x^2+2}$ 1

(iii) $y = \left(e^{2x}+3\right)^5$ 2

(iv) $y = x^3 e^{2x}$ 2

(v) $y = \dfrac{\log_e x}{x}$ 2

b)



The graphs of $y = x^2 - x - 6$ and $y = x + 2$ are shown on the diagram.

(i) Find the coordinates of P and Q. 3

(ii) Find the shaded area. 3

Question 2 – (14 marks) – Start a new page **Marks**

a) If $\log_a 2 = 1.31$ and $\log_a 3 = 2.07$ find the value of:

(i) $\log_a 6$ 1

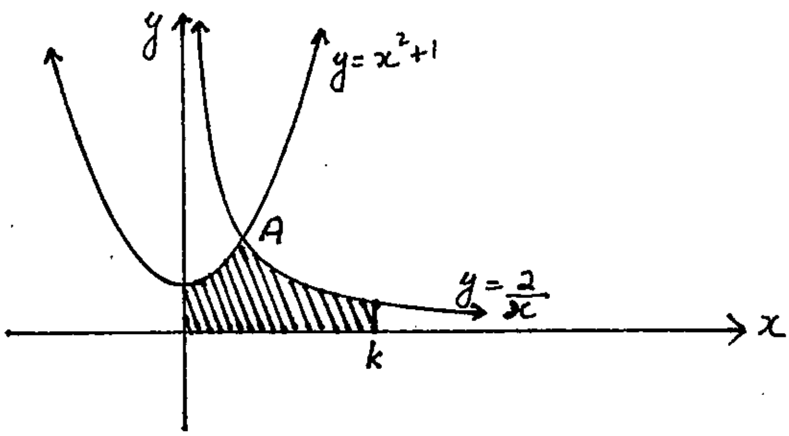
(ii) $\log_a 4.5$ 1

(iii) $\log_a \dfrac{8}{a^2}$ 2

b) Solve the equation $3^x = 17$, giving your answer correct to 3 decimal places. 2

c) Find the equation of the tangent to the curve $y = \log_e(3x-2)+4$ at the point where $x = 1$ 3

d)



The curves $y = x^2 + 1$ and $y = \dfrac{2}{x}$ meet at the point A, as shown on the diagram.

(i) Show that A has coordinates $(1, 2)$ 1

(ii) Find the value of k, given that the shaded area is $\dfrac{10}{3}$ units^2 4

Question 3 – (14 marks) – Start a new page **Marks**

a) Find the following:

(i) $\int x\sqrt{x}\,dx$ 1

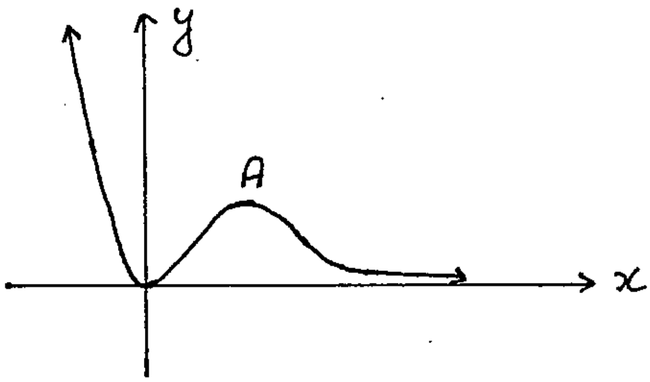
(ii) $\int \frac{1}{3x^2}\,dx$ 1

(iii) $\int x\,e^{x^2+1}\,dx$ 1

b) Show that $\int_{-6}^{-2} \frac{1}{x}\,dx = -\log_e 3$ 3

c) Use Simpson's Rule with 3 function values to find an approximate value of $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx$ correct to 3 decimal places. 3

d)



$y = x^2e^{-x}$ has a maximum turning point at A, as shown on the graph.

(i) Find the coordinates of A. 4

(ii) The equation $x^2e^{-x} - k = 0$ has 3 real, distinct roots. Using the graph, or otherwise, write down the possible values of k. 1

Question 4 – (14 marks) – Start a new page **Marks**

a) Solve $\log_3(x+3)+\log_3(x-5)=2$ 4

b) The curve $y=f(x)$ has a stationary point at the point $(2, 2)$.

If $\dfrac{d^2y}{dx^2}=\dfrac{2}{x^2}$ find $f(x)$ 4

c) The section of the curve $y=\dfrac{x}{\sqrt{x^3+1}}$ from $x=1$ to $x=3$ is rotated about the x axis. Find the volume of the solid of revolution. 3

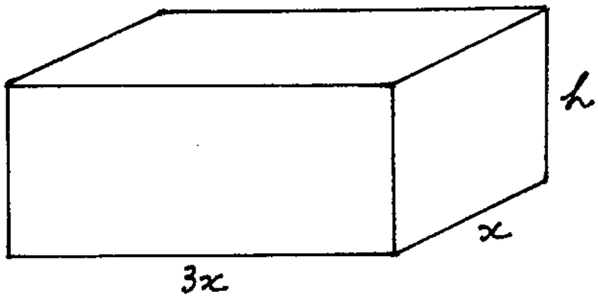
d) (i) Show that $f(x)=\dfrac{x^3}{x^2+1}$ is an odd function. 2

(ii) Hence, or otherwise, write down the value of $\displaystyle\int_{-2}^{2}\frac{x^3}{x^2+1}\,dx$ 1

Question 5 – (14 marks) – Start a new page **Marks**

a) Evaluate $\int_0^2 (3-2x)^3\,dx$ 2

b)



A rectangular prism has dimension x, $3x$ and h metres.

(i) If the surface area is 72m^2 show that $h = \dfrac{36-3x^2}{4x}$ 2

(ii) Hence show that the maximum volume of this prism is 36m^3 5

c) (i) Differentiate $y = (x+2)\sqrt{2x+1}$. Express your answer as a single fraction. 2

(ii) Hence or otherwise find $\int_4^{12} \dfrac{x+1}{\sqrt{2x+1}}\,dx$ 3

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Extension 1 Solutions

HSC Assessment Task I

Question 1

a.) (i) $y = \log_e(7x+6)$

$\frac{dy}{dx} = \frac{7}{7x+6}$

(ii) $y = e^{x^2+2}$

$\frac{dy}{dx} = 2x.e^{x^2+2}$

(iii) $y = (e^{2x}+3)^5$

$\frac{dy}{dx} = 5.(e^{2x}+3)^4.2e^{2x}$

$= 10e^{2x}(e^{2x}+3)^4$

(iv) $y = x^3e^{2x}$

$\frac{dy}{dx} = e^{2x}.(3x^2) + x^3.(2e^{2x})$

$= x^2e^{2x}(3+2x)$

(v) $y = \frac{\log_e x}{x}$

$\frac{dy}{dx} = \frac{x.\left(\frac{1}{x}\right) - \log_e x.(1)}{x^2}$

$= \frac{1-\log_e x}{x^2}$

b.) (i) $x^2 - x - 6 = x + 2$

$x^2 - 2x - 8 = 0$

$(x+2)(x-4) = 0$

$x = -2, 4$

when $x = -2$, $y = 0$

when $x = 4$, $y = 6$

$\therefore$ P is $(4,6)$ and Q is $(-2,0)$

(ii) $A = \int_{-2}^{4} x+2-(x^2-x-6)\, dx$

$= \int_{-2}^{4} (2x+8-x^2)\, dx$

$= \left[x^2 + 8x - \frac{x^3}{3}\right]_{-2}^{4}$

$= \left(16+32-\frac{64}{3}\right) - \left(4-16+\frac{8}{3}\right)$

$= 36 \text{ units}^2$

Question 2

a.) $\log_a 2 = 1.31$ $\quad \log_a 3 = 2.07$

(i) $\log_a 6 = \log_a(2\times3)$

$= \log_a 2 + \log_a 3$

$= 1.31 + 2.07$

$= 3.38$

(ii) $\log_a 4.5 = \log_a\left(\frac{3^2}{2}\right)$

$= 2\log_a 3 - \log_a 2$

$= 2\times2.07 - 1.31$

$= 2.83$

(iii) $\log_a \frac{8}{a^2} = \log_a 2^3 - \log_a a^2$

$= 3\log_a 2 - 2$

$= 3\times1.31 - 2$

$= 1.93$

b.) $3^x = 17$

$\log_e 3^x = \log_e 17$

$x = \frac{\log_e 17}{\log_e 3}$

$= 2.579$ (3 d.p.)

c.) $y = \log_e(3x-2)+4 \quad x=1$

$\frac{dy}{dx} = \frac{3}{3x-2}$

when $x=1$, $\frac{dy}{dx} = \frac{3}{1}$

when $x=1$, $y = \log_e 1 + 4$
$= 4$

$m = 1 \quad (1,4)$

$y - 4 = 1(x-1)$

$\therefore$ equation of the tangent is $y = x+3$

d.)(i) sub. $(1,2)$ in $y = x^2+1$

RHS $= 2$
$=$ LHS

sub. $(1,2)$ in $y = \frac{2}{x}$

RHS $= 2$
$=$ LHS.

$\therefore$ A has coordinates $(1,2)$ since it satisfies both equations.

(ii) $\frac{10}{3} = \int_0^1 x^2+1\,dx + \int_1^k \frac{2}{x}\,dx$

$= \left[\frac{x^3}{3}+x\right]_0^1 + \left[2\log_e x\right]_1^k$

$= \left(\frac{1}{3}+1-0\right) + (2\log_e k - 2\log_e 1)$

$= \frac{4}{3} + 2\log_e k$

$2\log_e k = 2$

$\log_e k = 1$

$\therefore k = e$

Question 3

a.) (i) $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx$

$= \frac{2x^{5/2}}{5} + c$

(ii) $\int \frac{1}{3x^2}\,dx = \int \frac{1}{3}x^{-2}\,dx$

$= \frac{1}{3}\cdot\frac{x^{-1}}{-1} + c$

$= \frac{-1}{3x} + c$

(iii) $\int x.e^{x^2+1}\,dx = \frac{1}{2}e^{x^2+1} + c$

b.) $\int_{-6}^{-2} \frac{1}{x}\,dx = \left[\log_e x\right]_{-6}^{-2}$

$= (\log_e -2 - \log_e$

$= \log_e\left(\frac{-2}{-6}\right)$

$= \log_e\left(\frac{1}{3}\right)$

$= \log_e 3^{-1}$

$= -\log_e 3$

c.) $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx$

x	0	1	2
$f(x)$	0.5	0.447	0.354

$\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx \approx \frac{2}{6}\left[f(0) + 4f(1) + f(2\right.$

$= \frac{1}{3}\left[0.5 + 4\times 0.447 + 0.3\right.$

$= 0.881$ (3 d.p.)

d.) $y = x^2 e^{-x}$

(i) $\frac{dy}{dx} = e^{-x}(2x) + x^2(-e^{-x})$

$= xe^{-x}(2-x)$

stat. pts occur when $\frac{dy}{dx} = 0$

$xe^{-x}(2-x) = 0$

$x = 0, 2$

when $x = 2$, $y = \frac{4}{e^2}$

$\therefore$ A has coordinates $\left(2, \frac{4}{e^2}\right)$

(ii) $0 < k < 2$

Question 4

a.) $\log_3(x+3) + \log_3(x-5) = 2$

$\log_3((x+3)(x-5)) = 2$

$(x+3)(x-5) = 3^2$

$x^2 - 2x - 15 = 9$

$x^2 - 2x - 24 = 0$

$(x+4)(x-6) = 0$

$x = -4, 6$

$\therefore x = 6$, since $x+3 > 0$ and $x-5 > 0$

b.) $\frac{d^2y}{dx^2} = \frac{2}{x^2}$ stat. pt. @ $(2,2)$

$\frac{dy}{dx} = \int 2x^{-2}\,dx$

$= -2x^{-1} + C_1$

when $\frac{dy}{dx} = 0$, $x = 2$

$\frac{dy}{dx} = -\frac{2}{x} + C_1$

$0 = -\frac{2}{2} + C_1$

$\therefore C_1 = 1$

$\frac{dy}{dx} = -\frac{2}{x} + 1$

$y = \int \left(-\frac{2}{x} + 1\right) dx$

$= -2\log x + x + C_2$

when $x = 2$, $y = 2$

$2 = -2\log 2 + 2 + C_2$

$\therefore C_2 = \log 4$

$\therefore f(x) = x - 2\log x + \log 4$

c.) $y = \frac{x}{\sqrt{x^3+1}}$ $x = 1$ to $x = 3$

$y^2 = \frac{x^2}{x^3+1}$

$V = \pi \int_1^3 \frac{x^2}{x^3+1}\,dx$

$= \frac{1}{3}\pi \left[\log(x^3+1)\right]_1^3$

$= \frac{\pi}{3}(\log 28 - \log 2)$

$= \frac{\pi}{3}\log 14$ units^3

d.) (i) $f(x) = \frac{x^3}{x^2+1}$

$f(-x) = \frac{-x^3}{x^2+1}$

$-f(x) = -\left(\frac{x^3}{x^2+1}\right)$

$= f(-x)$

$\therefore$ odd function

(ii) $\int_{-2}^{2} \frac{x^3}{x^2+1}\,dx = 0$

Question 5

a.) $\int_0^2 (3-2x)^3\,dx = \left[\frac{(3-2x)^4}{4\times -2}\right]_0^2$

$= \left[\frac{(3-2x)^4}{-8}\right]_0^2$

$= \frac{-1}{8} + \frac{81}{8}$

$= 10$

b.) (i) $SA = 2(3x \times x) + 2(3x \times h) + 2(x \times h)$

$= 6x^2 + 6xh + 2xh$

$= 6x^2 + 8xh$

$72 = 6x^2 + 8xh$

$8xh = 72 - 6x^2$

$h = \frac{72 - 6x^2}{8x}$

$\therefore h = \frac{36 - 3x^2}{4x}$

(ii) $V = 3x^2h$

$= 3x^2 \cdot \left(\frac{36 - 3x^2}{4x}\right)$

$= 27x - \frac{9x^3}{4}$

$\frac{dV}{dx} = 27 - \frac{27x^2}{4}$

stat. pt occurs when $\frac{dV}{dx} = 0$.

$27 = \frac{27x^2}{4}$

$x^2 = 4$.

$\therefore x = 2,\ x > 0$.

$\frac{d^2V}{dx^2} = -\frac{27x}{2}$

when $x = 2$, $\frac{d^2V}{dx^2} = -27 < 0$.

$\therefore$ max occurs when $x = 2$

when $x = 2$, $V = 27 \times 2 - \frac{9 \times 2^3}{4}$

$= 36\,m^3$

c.) (i) $y = (x+2)\sqrt{2x+1}$

$= (x+2)(2x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = (2x+1)^{1/2} \cdot 1 + (x+2) \cdot \frac{1}{2}(2x+1)^{-\frac{1}{2}} \cdot 2$

$= \sqrt{2x+1} + \frac{(x+2)}{\sqrt{2x+1}}$

$= \frac{2x+1+x+2}{\sqrt{2x+1}}$

$= \frac{3x+3}{\sqrt{2x+1}}$

(ii) $\int_4^{12} \frac{3x+3}{\sqrt{2x+1}}\,dx = (x+2)\sqrt{2x+1}$

$\therefore \frac{1}{3}\int_4^{12} \frac{3(x+1)}{\sqrt{2x+1}}\,dx = \int_4^{12} \frac{x+1}{\sqrt{2x+1}}\,dx$

$= \left[\frac{1}{3}(x+2)\sqrt{2x+1}\right]_4^{12}$

$= \frac{1}{3}(14 \times 5 - 6 \times 3)$

$= 17\frac{1}{3}$